Abstracting Shape Information from Point Clouds

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Objective

Provide a mathematical model of shape as absrtracted from three or higher dimensional point coulds. A shape model for point clouds should capture only the properties of the point clouds which are invariant under rotation scaling, translation and arbitrary permutations of the sequence of points in the cloud. We use the algebra of 3-tensors introduced by Mesner and P. Bhattacharya in [5,6,7] to investigate properties of the 3-uniform hypergraphs naturally deduced point clouds. Finally we propose to use as feature space for investigating properties of point clouds the spectral theory for tensors introduced by E. Gnang, A. Elgammal and V. Retakh in [1].

1 Tensors from point clouds and generalizing linear algebra

Given a finite sequence of points $\mathcal{P}$ in a three dimensional coordinate system which we refer to as a point cloud, we have

$$\mathcal{P} := (x_i,\, y_i,\, z_i)_{0 \le i < n} \tag{1}$$

We therefore note that points clouds are no more than a collection of real value ordered triplets. By viewing the real values which appear as coordinates of points as abstract vertex labels, we are naturally led to assoaciate to our point cloud a directed 3-uniform hypergraph. As typically done with, graphs we associate with the deduced directed 3-uniform hypergraphs an adjacency 3-tensor.

Recall that a 3-tensor $\boldsymbol{A}$ of dimensions $m \times n \times p$ denotes a rectangular cuboid array of numbers. The array consists of m rows, n columns, and p depths with the entry $a_{i,j,k}$ occupying the position where the i^{th} row, the j^{th} column, and the k^{th}depth meet. For many purposes it will suffice to write

$$\boldsymbol{A} = \left(a_{i,j,k}\right)\ (1 \le i \le m;\, 1 \le j \le n;\, 1 \le k \le p)\,, \tag{2}$$

1.1 A ternary product

Ternary product of tensors: First proposed by Mesner and P. Bhattacharya in [5,6,7] as a generalization of matrix multiplication. we recall that $\boldsymbol{A} = \left(a_{i,j,k}\right)$ be a tensor of dimensions $(m \times l \times p)$, $\boldsymbol{B} = \left(b_{i,j,k}\right)$ a tensor of dimensions $(m \times n \times l)$, and $\boldsymbol{C} = \left(c_{i,j,k}\right)$ a tensor of dimensions $(l \times n \times p)$; the ternary product of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ results in a tensor $\boldsymbol{D} = \left(d_{i,j,k}\right)$ of dimensions $(m \times n \times p)$ denoted

$$\boldsymbol{D} = \circ\left(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}\right) \tag{3}$$

and the product is expressed by :

$$d_{i,j,k} = \sum_{1 \le t \le l} a_{i,t,k} \cdot b_{i,j,t} \cdot c_{t,j,k} \tag{4}$$

1.2 Tensor orthogonality

We recall from linear algebra that orthogonality is defined by

$$\boldsymbol{Q} \cdot \boldsymbol{Q}^{\dagger} = \boldsymbol{\Delta}. \tag{5}$$

Similarly 3-tensor orthogonality can be defined by

$$\circ\left(\boldsymbol{Q}, \boldsymbol{Q}^{\dagger^2}, \boldsymbol{Q}^{\dagger}\right) = \boldsymbol{\Delta} \tag{6}$$

2 3-Tensor spectrum as feature space for point clouds

The spectral decomposition for diagonalizable matrices can be expressed as the existence of solutions to the system of equations

$$\begin{cases} \boldsymbol{A} & = (\boldsymbol{D}\boldsymbol{Q})^{\dagger}\,(\boldsymbol{E}\boldsymbol{R}) \\ \boldsymbol{\Delta} & = \quad \boldsymbol{Q}^{\dagger}\boldsymbol{R} \\ \boldsymbol{D}^{\dagger} \star \boldsymbol{E} & = \quad \boldsymbol{D}^{\dagger}\boldsymbol{E} \end{cases}, \tag{7}$$

where $\star$ denote the Hadamard product.

Theorem 2.1 (Spectral Theorem for 3-Tensors) *For an arbitrary hermitian non zero 3-tensor $\boldsymbol{A}$ with $\|\boldsymbol{A}\|_{\ell_3}^{3} \neq 1$ there exist a factorization of the form:*

$$\begin{cases} \boldsymbol{A} = \circ\left(\circ\left(\boldsymbol{Q}, \boldsymbol{D}, \boldsymbol{D}^{T}\right), \left[\circ\left(\boldsymbol{R}, \boldsymbol{E}, \boldsymbol{E}^{T}\right)\right]^{\dagger^2}, \left[\circ\left(\boldsymbol{S}, \boldsymbol{F}, \boldsymbol{F}^{T}\right)\right]^{\dagger}\right) \\ \boldsymbol{\Delta} = \qquad \circ\left(\boldsymbol{Q},\, \boldsymbol{R}^{\dagger^2},\, \boldsymbol{S}^{\dagger}\right) \end{cases} \tag{8}$$

where $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$ denote scaling tensors.

For convenience we introduce the following notation for scaled tensors

$$\begin{cases} \widetilde{\boldsymbol{Q}} = \circ\left(\boldsymbol{Q}, \boldsymbol{D}, \boldsymbol{D}^{T}\right) \\ \widetilde{\boldsymbol{R}} = \circ\left(\boldsymbol{R}, \boldsymbol{E}, \boldsymbol{E}^{T}\right) \\ \widetilde{\boldsymbol{S}} = \circ\left(\boldsymbol{S}, \boldsymbol{F}, \boldsymbol{F}^{T}\right) \end{cases} \tag{9}$$

and simply express the tensor decomposition of $\boldsymbol{A}$ as:

$$\boldsymbol{A} = \circ\left(\widetilde{\boldsymbol{Q}}, \widetilde{\boldsymbol{R}}^{\dagger^2}, \widetilde{\boldsymbol{S}}^{\dagger}\right) \tag{10}$$

2.1 The Spectrum of n-tensors.

In order to formulate the spectral theorem for $\boldsymbol{A} \in \mathbb{C}^{l^n}$ we will briefly discuss the notion of orthogonal n-tensors, which can be expressed as

$$\boldsymbol{\Delta} = \bigcirc_{t=1}^{n}\left(\boldsymbol{Q}^{\dagger^{(n+1-t)}}\right) \tag{11}$$

that is

$$\delta_{i_1,i_2,\cdots,i_n} = \sum_{k}\left(\left(\prod_{t=1}^{n-1} q^{\dagger^{(n+1-t)}}_{i_1,i_2,\cdots,i_t,\,k\,,i_{t+2}\cdots,\,i_n}\right) q^{\dagger}_{k,\,i_2,\cdots,i_n}\right), \tag{12}$$

Where $\dagger$ denotes the generalized transpose conjugate operation, which still corresponds to a cyclic permutation of the indices.

We first provide the formula for the scaling tensor whose product with $\boldsymbol{A}$ leaves the tensor unchanged.

$$a_{i_1,i_2,\cdots,i_n} = \left(\bigcirc\left(\boldsymbol{A}, \boldsymbol{D}^{(1)}, \boldsymbol{D}^{(2)}, \boldsymbol{D}^{(3)}, \cdots, \boldsymbol{D}^{(n-1)}\right)\right)_{i_1,i_2,\cdots,i_n} \tag{13}$$

$$\Rightarrow \begin{cases} \forall t < n-2 \quad \boldsymbol{D}^{(t)} \equiv \left(d^{(t)}_{i_1,i_2,\cdots,i_n} = \delta_{i_2,i_{2+t}}\right) \\ \qquad \boldsymbol{D}^{(n-1)} \equiv \left(d^{(n-1)}_{i_1,i_2,\cdots,i_n} = \delta_{i_1,i_2}\right) \end{cases} \tag{14}$$

The general scaling tensors are expressed by

$$\begin{cases} \forall t < n-2 \quad \boldsymbol{S}^{(t)} \equiv \left(s^{(t)}_{i_1,i_2,\cdots,i_n} = \delta_{i_2,i_{2+t}} \cdot \omega_{i_t,i_{2+t}}\right) \\ \qquad \boldsymbol{S}^{(n-1)} \equiv \left(s^{(n-1)}_{i_1,i_2,\cdots,i_n} = \delta_{i_1,i_2} \cdot \omega_{i_1,i_{n-1}}\right) \end{cases} \tag{15}$$

where $\boldsymbol{W} = \left(w_{m,n}\right)$ is a symmetric matrix. The expression for the scaled orthogonal tensor is therefore expressed by

$$\left(\bigcirc\left(\boldsymbol{Q}, \boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)}, \boldsymbol{S}^{(3)}, \cdots, \boldsymbol{S}^{(n-1)}\right)\right)_{i_1,i_2,\cdots,i_n} = q_{i_1,i_2,\cdots,i_n}\left(\prod_{k \neq 2} \omega_{i_2,i_k}\right) \tag{16}$$

From which it follows that the scaled tensor which will be of the form:

$$\widetilde{\boldsymbol{Q}} = \bigcirc\left(\boldsymbol{Q}, \boldsymbol{S}^{(1)}, \boldsymbol{S}^{(2)}, \boldsymbol{S}^{(3)}, \cdots, \boldsymbol{S}^{(n-1)}\right) \tag{17}$$

Theorem 2.2: *(Spectral Theorem for n-Tensors):* For any non zero hermitian tensor $\boldsymbol{A} \in \mathbb{C}^{l^n}$ such that $\|\boldsymbol{A}\|_{\ell_n}^{n} \neq 1$, there exist a factorization in the form

$$\begin{cases} \boldsymbol{A} = \bigcirc_{t=1}^{n}\left(\widetilde{\boldsymbol{Q}}_t^{\dagger^{(n+1-t)}}\right) \\ \boldsymbol{\Delta} = \bigcirc_{t=1}^{n}\left(\boldsymbol{Q}_t^{\dagger^{(n+1-t)}}\right) \end{cases} \tag{18}$$

the expression above generalizes Eq[10]

3 Polynomial formalization of shape from point coulds

We formalize the notion of shape in two steps. First we use polynomials to describe our adjacency tensors and call the resulting polynomial the adjacency polynomial. The relevent domain for the adjacency polynomial will be conveniently chosen to be the n-th root of unity. Finally, we define the shape of the point cloud in terms of invariance properties of the computed adjacency polynomial.

Consider the parametric family of polynomial rings defined for some arbitrary field F and integers $m, n, p > 0$ by

$$F^{x,y,z}_{(m,n,p)} := \left(\left(^{F[x]}/_{x^m-1}\right)[y]/_{(y^n-1)}\right)[z]/(z^p-1). \tag{19}$$

As a consequence of the *Lagrange interpolation formula* it follows that $\forall\, f \in \mathbb{C}\,[x,\, y,\, z]$, the unique minimal degree polynomial element of $\mathbb{C}^{x,y,z}_{(m,n,p)}$ which is congruent to f is expressed by

$$\sum_{(r_0,r_1,r_2)\in\Omega_m\times\Omega_n\times\Omega_p} f\left(r_0, r_1, r_2\right)\left(\prod_{s_0\in\Omega_m\backslash\{r_0\}}\left(\frac{x-s_0}{r_0-s_0}\right)\right) \times \left(\prod_{s_1\in\Omega_n\backslash\{r_1\}}\left(\frac{y-s_1}{r_1-s_1}\right)\right) \times \left(\prod_{s_2\in\Omega_p\backslash\{r_2\}}\left(\frac{z-s_2}{r_2-s_2}\right)\right) \tag{20}$$

where

$$f\left(r_0, r_1, r_2\right) := f\left(t_0, t_1, t_2\right) \mod \begin{Bmatrix} t_0 - r_0 \\ t_1 - r_1 \\ t_2 - r_2 \end{Bmatrix}. \tag{21}$$

We note that the degrees of freedom in this encoding corresponds precisely to that of an $m \times n \times p$ 3-tensors. Having used polynomials to model the adjacency structure of our hypergraph, we now use polynomials to model shape. Given an addjacency polynomial $f \in \mathbb{C}\,[x, y, z]$ we define the induced shape over $\Omega_n \times \Omega_n \times \Omega_n$ to be expressed by

$$f(p(x), p(y), p(z)) \mod \left(t^n - 1\right) - \prod_{r\in\Omega_n}\left(t - p(r)\right) \tag{22}$$

Conclusion and future work

Using tensor spectral analysis we have been able to model the shape content of clouds of points. Our model for shape is provably robust to rotation scaling, translation and arbitrary permutation of the points in the sequence. In our future work we hope to implement fast libraries for using the our shape model for object recognition tasks in computer vision.

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